PhD Qualifying Examination: Classical Electrodynamics

Instruction: Each question carries equal 25 marks, the marks for individual parts are marked in bold faced box [...], please state your calculations clearly.

1. • Starting with electrostatic potential for a charge distribution $\rho(\mathbf{r})$:

$$\varphi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \tag{1}$$

to prove Green's reciprocal relation [**5**]:

$$\int d^3r \rho_1(\mathbf{r})\varphi_2(\mathbf{r}) = \int d^3r \rho_2(\mathbf{r})\varphi_1(\mathbf{r}). \tag{2}$$

Then use it to prove that an empty sphere with charge spread uniformly over its surface to prove this "mean value theorem":

$$\frac{1}{4\pi R^2} \int dS \varphi(\mathbf{r}) = \varphi(0), \tag{3}$$

the average of $\varphi(\mathbf{r})$ over a spherical surface S that encloses a charge-free volume is equal to the potential at the center of the sphere. [**10**]

• Show that we can also obtain same mean value theorem above by using Green's second identity in vector calculus:

$$\int_V d^3r (f\nabla^2 g - g\nabla^2 f) = \int_S d\mathbf{S} \cdot (f\nabla g - g\nabla f) \tag{4}$$

where f, g are arbitrary functions. [**5**]

• Using the result above to give an alternative derivation of Earnshaw's theorem: The scalar $\varphi(\mathbf{r})$ in a finite, charge free region of space R takes its maximum/minimum on the boundary of R. [**5**].

2. • Use the formula for electrostatic total energy:

$$U_E = \frac{1}{2} \int d^3r \rho(\mathbf{r})\varphi(\mathbf{r}) \tag{5}$$

to find the interaction energy V_E between two identical insulating spheres of radius R and charge Q distributed uniformly over their surfaces. Their center to center separation is $d > 2R$. Comment on the dependence of V_E on R. [**8+2**]

• Instead of being Coulombic, suppose the electrostatic potential is produced by:

$$\varphi(\mathbf{r}) = \frac{q}{4\pi\epsilon_0} f(|\mathbf{r}|). \tag{6}$$

  – Calculate the potential produced by an infinite flat sheet at $z = 0$ with uniform charge/area σ. [**10**]

– Show that the associated electric field is [**5**]:

$$\mathbf{E}(z) = \frac{\sigma}{2\epsilon_0} z f(z) \hat{\mathbf{z}}. \tag{7}$$

3. (a) Apply Green's second identity:

$$\int_V d^3x \left(\phi \nabla^2 \psi - \psi \nabla^2 \phi \right) = \oint_{S=\partial V} d^2x \, \hat{\mathbf{n}} \cdot \left(\phi \nabla \psi - \psi \nabla \phi \right) \tag{8}$$

and set $\phi = G(\mathbf{y}, \mathbf{x})$ and $\psi = G(\mathbf{z}, \mathbf{x})$. Using $\nabla_{\mathbf{x}}^2 G(\mathbf{y}, \mathbf{x}) = -\delta^{(3)}(\mathbf{x} - \mathbf{y})$ to express the difference $G(\mathbf{y}, \mathbf{z}) - G(\mathbf{z}, \mathbf{y})$ in terms of an integral over the surface $S = \partial V$. [**5**]

(b) Show that a Green's function $G_D(\mathbf{x}, \mathbf{y})$ with Dirichlet boundary conditions $G_D(\mathbf{x}, \mathbf{y}) = 0$ for $\mathbf{y} \in \partial V$ must be symmetric in $\mathbf{x}$ and $\mathbf{y}$. [**5**]

(c) Argue that $\mathbf{n_y} \cdot \nabla_{\mathbf{y}} G_D(\mathbf{x}, \mathbf{y}) \to -\delta^2(\mathbf{x} - \mathbf{y})$ for $\mathbf{x} \to \partial V$ and $\mathbf{y} \in \partial V$. For the case $\mathbf{x} \neq \mathbf{y}$, you can use the Dirichlet boundary condition for $G_D(\mathbf{x}, \mathbf{y})$. To understand the special case $\mathbf{x} \to \mathbf{y}$, integrate the above expression over all $\mathbf{y} \in \partial V$ before performing the limit.[**5**]

(d) Consider the Neumann boundary condition $\mathbf{n_y} \cdot \nabla_{\mathbf{y}} G_N(\mathbf{x}, \mathbf{y}) = -F(\mathbf{y})$ for $\mathbf{y} \in \partial V$ with $\oint_{\partial V} d^2x \, F(\mathbf{x}) = 1$. Show that $G_N(\mathbf{x}, \mathbf{y})$ is not symmetric in $\mathbf{x}$ and $\mathbf{y}$ in general. Construct a Green's function $G_N'(\mathbf{x}, \mathbf{y}) = G_N(\mathbf{x}, \mathbf{y}) + H(\mathbf{y}) + K(\mathbf{x})$ that is symmetric in $\mathbf{x}$ and $\mathbf{y}$. What properties must H and K have? [**4**]+[**4**]+[**2**]

4. It is true (but not obvious) that any vector field $\mathbf{V}(\mathbf{r})$ which satisfies $\nabla \cdot \mathbf{V}(\mathbf{r}) = 0$ can be written *uniquely* in the form

$$\mathbf{V}(\mathbf{r}) = \mathbf{T}(\mathbf{r}) + \mathbf{P}(\mathbf{r}) = \mathbf{L}\psi(\mathbf{r}) + \nabla \times \mathbf{L}\gamma(\mathbf{r}),$$

where $\mathbf{L} = -i\mathbf{r} \times \nabla$ is the angular momentum operator and $\psi(\mathbf{r})$ and $\gamma(\mathbf{r})$ are scalar fields. $\mathbf{T}(\mathbf{r}) = \mathbf{L}\psi(\mathbf{r})$ is called a *toroidal field* and $\mathbf{P}(\mathbf{r}) = \nabla \times \mathbf{L}\gamma(\mathbf{r})$ is called a *poloidal field*. This decomposition is widely used in laboratory plasma physics.

(a) Confirm that $\nabla \cdot \mathbf{V}(\mathbf{r}) = 0$. [**5**]

(b) Show that a poloidal current density generates a toroidal magnetic field and vice versa. [**5**]

(c) Show that $\mathbf{B}(\mathbf{r})$ is toroidal for a toroidal solenoid. [**5**]

(d) Suppose there is no current in a finite volume V. Show that $\nabla^2 \mathbf{B}(\mathbf{r}) = 0$ in V. [**5**]

(e) Show that $\mathbf{A}(\mathbf{r})$ in the Coulomb gauge is purely toroidal in V when $\psi(\mathbf{r})$ and $\gamma(\mathbf{r})$ are chosen so that $\nabla^2 \mathbf{B}(\mathbf{r}) = 0$ in V. [**5**]