QUALIFIER
QUANTUM II

Data provided: As needed

DEPARTMENT OF PHYSICS NTU **February 2025**

QUANTUM MECHANICS II

2 hours

Date of Exam:
Examiner:

Answer all questions.
All question marks summed together, equal 100 points.

Question 1. (Time dependent perturbation theory)

Consider the following Hamiltonian

$$H = H_0 + H'(t) = H_0 + Uexp\left(-\frac{t}{T}\right)$$

where U is a time independent operator and T a (decay) constant.
Show that the probability to lowest order in U that the perturbation will produce a transition from one eigenstate n of H_0 to a different eigenstate m of H_0, during a time interval from $t = 0$ to a time $t \gg T$ is given by:

$$|c_m(t)|^2 \xrightarrow{t \gg T} \frac{|\langle m|U|n\rangle|^2}{(E_m - E_n)^2 + \frac{\hbar^2}{T^2}}$$

[15]

Question 2. (perturbation theory)

A simple one-dimensional harmonic oscillator is subjected to a perturbing linear potential $\widehat{H}_I = b\hat{x}$, where b a real number, and $\hat{x}$ the position.
(a) Calculate the energy shift to the first non-vanishing order of perturbation theory. [10]
(b) Solve the problem exactly and compare with the result obtained in question (a). [10]
Useful: $\langle n'|\hat{x}|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n+1}\delta_{n',n+1} + \sqrt{n}\delta_{n',n-1}\right)$

Question 3. (Scattering)

Use the optical theorem to show that there is an upper bound on the elastic scattering cross section $\sigma_{scattering}$ in the case where the scattering amplitude $f_k(\theta, \phi)$ is independent of θ and ϕ.
(Hint: Optical theorem gives $\sigma_{scattering} = \frac{4\pi}{k} Im[f_k(0)]$). [15]

Question 4. (Scattering)

Start with the equation for l-scattering cross section

$$\sigma_l = 4\pi|f_l(k)|^2(2l+1) = 4\pi(2l+1)\left[\frac{1}{k\cot\delta_l - ik}\right]^2$$

to show that:

$$\sigma_l = \frac{4\pi}{k^2}(2l+1)\frac{(\Gamma/2)^2}{(E-E_R)^2 + (\Gamma/2)^2}$$

where for simplicity we can take $k = E$, and E_R is the resonance energy. Here the resonance width Γ, is defined as follows: $\frac{1}{\Gamma} = \frac{1}{2}\frac{d\delta_l}{dE}$ [15]

Question 5. (Scattering)

In a scattering experiment where a spinless non-relativistic particle of mass m is scattered by an unknown potential, a resonance is observed where the cross section has a maximum at energy $E = E_R$. Use the data of this experiment that are described by the Breit-Wigner formula:

$$\sigma_{scat} = \frac{\pi(2l+1)}{k^2}\frac{\Gamma^2}{(E-E_R)^2+\frac{\Gamma^2}{4}}$$

to find a value for the orbital angular momentum l of the resonance. [15]

Question 6. (Green's functions)

The unperturbed Schrödinger equation has the form ($\hbar = 1$):

$$\left(i\frac{\partial}{\partial t} + \frac{\nabla^2}{2m}\right)\psi(\vec{x},t) = 0 \qquad (1)$$

with a Kernel function $K(\vec{x},t;\vec{y},t')$ satisfying the homogeneous equation:

$$\left(i\frac{\partial}{\partial t} + \frac{\nabla^2}{2m}\right)K(\vec{x},t;\vec{y},t') = 0 \qquad (2)$$

with the initial condition:

$$K(\vec{x},t;\vec{y},t) = \delta^{(3)}(\vec{x}-\vec{y}) \qquad (2a)$$

If we impose the boundary condition that $K(\vec{x},t;\vec{y},t)$ vanishes for $t' < t$, we obtain the (*retarded*) Green's function:

$$G_+(\vec{x},t;\vec{y},t') = \Theta(t-t')K(\vec{x},t;\vec{y},t') \qquad (3)$$

satisfying the inhomogeneous equation:

$$\left(i\frac{\partial}{\partial t} + \frac{\nabla^2}{2m}\right)G_+(\vec{x},t;\vec{y},t') = i\delta(t-t')\delta^{(3)}(\vec{x}-\vec{y}) \qquad (4)$$

Show that the Green's function (3) satisfies the inhomogeneous equation (4). [10]

Question 7. (Lippmann-Schwinger)

The "in" states in a scattering problem satisfy the time independent Schrödinger equation:

$$(E - H_0)|\psi_\alpha\rangle^{(+)} = V|\psi_\alpha\rangle^{(+)}$$

show by rewriting the Schrödinger equation above, that a formal solution of the Schrödinger equation is the Lippmann-Schwinger equation:

$$|\psi_\alpha\rangle^{(+)} = |\varphi\rangle + \frac{1}{E-H_0+i\epsilon}V|\psi_\alpha\rangle^{(+)},$$

where $|\varphi\rangle$ is the initial state of $|\psi_\alpha\rangle^{(+)}$ very far from the potential: $\lim_{t\to-\infty}|\psi_\alpha\rangle^{(+)} = |\varphi\rangle$ where the potential is negligible.

In this solution we have the presence of an $i\epsilon$ factor, where ϵ a real positive number. Why is this imaginary shift (analytic continuation) in energy introduced? Why is this particular choice used? Why not $-i\epsilon$? Hint: in your argument you may want to use the time evolution operator $U(t, t_0) = exp\big(-iE(t - t_0)\big)$. [10]

END OF EXAMINATION PAPER