Spring 2025

Qualifying Exam — Quantum Mechanics I

Instructions: Define or explain clearly your symbols and notations. The score is indicated in front of each subproblem.

1. Consider uncertainty relation for the one-dimensional position and momentum observables in quantum mechanics.

   (a) [10%] Define mathematically the uncertainties involved here.

   (b) [10%] State mathematically the uncertainty relation and explain its physical meaning.

   (c) [10%] Prove the above uncertainty relation.

2. Consider the one-dimensional simple harmonic oscillator of mass m given by the Hamiltonian

$$\mathcal{H} = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 X^2 = \left(a^\dagger a + \frac{1}{2}\right)\hbar\omega = \left(N + \frac{1}{2}\right)\hbar\omega \ ,$$

   where

$$a \equiv \frac{1}{\sqrt{2}}\left(\frac{X}{x_0} + i\frac{x_0 P}{\hbar}\right) \ , \quad N \equiv a^\dagger a \ , \ \text{and} \quad x_0 = \sqrt{\frac{\hbar}{m\omega}} \ .$$

   Denote the set of energy eigenstates by $\{|n\rangle\}$ with $N\,|n\rangle = n\,|n\rangle$.

   (a) [10%] Start from the canonical commutation relation of X and P, derive the following commutation relations:

$$[a, a^\dagger] \ , \ [a, N] \ , \ \text{and} \ [a, \mathcal{H}].$$

   (b) [10%] Work out the matrix representations of the following operators: a, $\mathcal{X}$, and $\mathcal{P}$. You are required to show explicitly the matrix elements of the upper-left 4×4 sub-matrix.

   (c) [10%] What transitions between *different* levels can be induced by the X and X^2 operators? What are the corresponding energies involved in the transitions of the precious problem?

3. Consider spin-1/2 systems and use the S_z basis for the following subproblems. Define the polarization vector operator $P_i = \sigma_i$, the Pauli matrices.

   (a) [10%] For an ensemble of states polarized in the $\hat{\mathbf{n}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ direction, compute its density matrix ρ. Also, write your result in terms of a linear combination of $\mathbb{1}$ and the Pauli matrices $\boldsymbol{\sigma}$.

   (b) [10%] For the above subproblem, compute the ensemble average of the polarization vector, $[\mathbf{P}]$.

   (c) [10%] Consider a mixed ensemble. Suppose the ensemble averages $[S_x]$, $[S_y]$ and $[S_z]$ are all known. Show how we may construct the density matrix that characterizes the ensemble.

   (d) [10%] Suppose an ensemble with 50% of the particles polarized in the $+z$ direction and the other 50% of the particles polarized in a direction that lies on the x-z plane and makes an angle α with the $+z$ axis. What is the density operator ρ that describes this ensemble?